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AXIOMATIC APPROACH TO RANKINGS TECHNIQUES OF DECISION ANALYSIS

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ABSTRACT

Decision-making constitutes an integral part of human life, encompassing daily activities such as shopping and travel planning, as well as political elections. The decision-making process relies on the analysis of available options based on various criteria, enabling hierarchical ordering and the selection of the optimal alternative. In the case of decisions with long-term consequences, such as choosing the location of a production plant or investment strategy, spontaneity is unacceptable. With the increasing availability of information, the necessity of considering numerous potential options becomes a challenge.

In decision-making theory, various methods for evaluating objects have been developed, categorized as methods of total order or partial order, aligning with the mathematical concept of linear order. There are many natural, intuitive and desired properties of ranking techniques of multi-criteria decision-making. These properties can be expressed in terms of functional equations and inequalities. In such setting, the desired properties can be investigated with straightforward proof. With an approach of the functional equations and inequalities, ranking techniques can be evaluated in terms of the desired properties what enables a choice of an optimal ranking method for a given task.

The article presents a short review of ranking techniques of multi-criteria decision-making. It makes conclusions about the common ideas shared among most of presented ranking techniques. In final, four properties of selected ranking techniques are investigated, namely: symmetry, scale-invariance, shift-invariance, and boundness.

Key words: Multi-criteria decision making, Linear ordering, Ranking techniques, Functional equations, Functional inequalities.

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1. Introduction

In the recent years, decision-making process is more and more focused on time-effective and cost-effective decisions. It has various causes and may be related to broader social, technological, and cultural changes. It is particularly associated with efficiency and the need to achieve goals in the business world but leaves its mark on all aspects of human activity. During the decision-making process, we analyze available options based on various criteria, informally evaluating them. This allows us to organize possibilities from best to worst and make a choice. In many situations, spontaneous decisions are not feasible. For example, choosing the location of a manufacturing plant, supplier, or investment strategy may have long-term consequences. Furthermore, the increase in the amount of available information necessitates considering a greater number of potential options and their parameters in the decision-making process. We can distinguish several approaches to the decision-making process (Figueira,2005):

- Multi-Attribute Utility Theory (MAUT) (Von Winterfeldt, & Edwards, 1986): this approach involves assigning weights to different criteria and aggregating them into an overall utility score to compare alternatives.
- Analytic Hierarchy Process (AHP) (Saaty, 1994): this approach entails decomposing a complex decision problem into a hierarchy of criteria and alternatives, enabling decision-makers to assess and prioritize based on pairwise comparisons.
- ELimination and Choice Expressing REality (Elimination Et Choix Traduisant la Réalité) – ELECTRE (Roy,1996): this is a family of multi-criteria decision-making methods developed by French researcher Bernard Roy in the 1960s. It is a relatively popular tool in decision analysis, with various variants and extensions. Essentially, it involves evaluating alternatives based on their concordance and discordance with pre-defined criteria.

In the field of decision-making theory, numerous techniques for evaluating objects described by multiple parameters have been developed. These techniques can essentially be classified as methods of total order or partial order, directly related to the mathematical concepts of linear and partial order. Methods of total order provide a single real number for each analyzed object, typically interpreted as "the more, the better," naturally introducing a total order to the set of alternatives.

Many proposed evaluation techniques have been developed based on the researcher's intuition regarding desirable statistical properties. The choice of an evaluation technique constitutes an additional challenge that decision-makers must confront.

In this paper, we present an axiomatic approach to ranking based on natural postulates associated with data analysis. Axiomatic approach to decision-making process taking into account utility of alternatives has been introduced in the beginning of 20th century. The classical and widely-known utility model, namely model of the expected utility, was introduced in von Neumann, Morgenstern (1995). Their approach was based on axiomatic formulation of properties of the preference relation, namely: translativity, completeness, independence and continuity. Further, Quiggin (1982) using axiomatic approach formulated the rank-dependent utility model. Kahneman and Tversky in their joint works from 1979 and 1992 developed the prospect theory and the cumulative prospect theory. Such approach in the utility theory was used in works e.g. Schmeidler (1989), Prelec (1998), Axiomatic approach is getting more appeal also in other statistical areas. Atkinson (Atkinson (1970)) proposed an inequality measure which was derived from six axioms. These works were motivation to make a research on axiomatic approach to rankings.

Evaluation techniques for objects often prove effective in conducting linear ordering of objects, essentially yielding a ranking of these objects. We start with overview of ranking techniques. It turns out that most of them share the same desired properties stemming from postulates of data analysis. Further, we formulate these properties in a language of functional equation and inequalities. Finally, we investigate some very natural and desired properties of ranking techniques, that is symmetry, scale-invariance, shift-invariance, and boundness for three techniques: Simple Additive Ranking, Absolute Reference, and TOPSIS.

2. Overview of Ranking Methods

Ranking is a broader concept than linear ordering, which involves establishing a certain way of arranging elements of a set so that for any two different elements, it is possible to determine which one is greater, smaller, or equal to the other.

In mathematics, linear ordering is often used concerning ordered sets, for example, in real numbers, where their values can be compared, determining which is greater than the other.

The origins of mathematical linear ordering can be observed in the achievements of ancient civilizations such as the Babylonians and ancient Egyptians, who used simple methods to organize and order numerical data.

One of the key moments in the development of mathematical linear ordering was the introduction of the order axiom by the Greek mathematician Euclid, allowing for the formal definition of the order relation on numbers.

The development of algebraic methods for linear ordering significantly accelerated during the Middle Ages, particularly thanks to the work of Arab mathematicians who devised techniques for solving linear equations and inequalities.

Today, mathematical linear ordering is an integral part of many fields such as mathematical analysis, graph theory, and statistics, where it enables data ordering and analysis of relationships between them. The advancement of computer science and algorithmics has also contributed to the development of mathematical linear ordering by creating effective methods for sorting data, which are widely used in modern computer systems.

Ranking methods can be categorized into several groups. Such distinction may be derived from

- the way what scale is used to treat decision-making criteria
- symmetric or asymmetric treating decision-making criteria

Rankings may be based on the ordinal scale. This group includes concordance analysis, dominance functions, and Hasse average ranking (Brüggemann, et al., 2005). These methods do not take into account exact difference between values of decision-making criterium. If the exact difference is essential to know, there are methods assuming interval or ratio scale of each decision-making criterium. Within that group, there are methods based on the assumption that for each variable, we individually select normalizing function $f_j: \mathbb{R} \to [0,1]$, which may generally differ between variables. Consequently, we cannot change the order of input variables. Ultimately, arithmetic or geometric mean, possibly considering variable weighting, is used for ranking. Examples of such methods include desirability functions and utility functions (Kahneman & Tversky, 1979).

Finally, there are also methods which involve the same normalizing function for each decision-making criterium. Among them, we can mention methods such as Simple Additive Ranking (SAR), Absolute Reference, TOPSIS (Zeleny,1982). These methods, along with selected properties, are the subject of analysis in this article.

Let us recall some ranking techniques. Zdzisław Hellwig, the head of the Department of Statistics at the former Wrocław University of Economics, was one of the pioneers in introducing linear ordering. The method he proposed allows for establishing ranking of objects described in a multidimensional space of features, taking into account certain ordering criteria. Hellwig introduced key terms such as stimulants and destimulants, and proposed two variants of the method: Simple Additive Ranking, Absolute Reference also known as the method not utilizing a developmental pattern and the method utilizing a developmental pattern (Hellwig, 1968).

Let us assume we have n objects (indexed by i) and k variables (indexed by j). We may classify these variables into two groups:

- Stimulant – a variable for which high values are favorable for the phenomenon under study (the higher the value of this feature, the better).
- Destimulant – a variable for which low values are favorable for the phenomenon under study (the lower the value of this feature, the better).

The process of determining the ranking (ordering of multi-dimensional objects) using the non-reference Hellwig method (absolute reference) begins with normalization to unify measurement units and scales. Typically, the method employs the transformation in the form of:

$$z_{ij} = \frac{x_{ij} - \min(x_j)}{\max(x_j) - \min(x_j)} \quad (1)$$

for stimulants

$$z_{ij} = \frac{\max(x_j) - x_{ij}}{\max(x_j) - \min(x_j)} \quad (2)$$

for destimulants, where x_{ij} is a value of j-th variable for i-th feature.

Subsequently, it is necessary to calculate, for each object, the arithmetic mean of the transformed feature values. Based on these calculations, the objects are ordered (creating a ranking).

$$R_i = \sum_{j}^{n} \frac{z_{ij}}{n}.$$

The above formula can be modified using assigned weights to variables. In such a case, one should use a weighted average. The second linear ordering method by Hellwig utilizes standardization for normalization according to the formula:

$$z_{ij} = \frac{x_{ij} - \overline{x_j}}{\sigma_j} \quad (3)$$

for the stimulant set, where $\overline{x_j}$ is the mean of values and σ_j is the standard deviation of the j-th variable,

$$z_{ij} = -\frac{x_{ij} - \overline{x_j}}{\sigma_j} \quad (4)$$

for the destimulant set.

After this transformation, each variable also takes on the nature of a stimulant. The next step involves determining the reference point $L_j = \max_i z_{ij}$.

Subsequently, for each object, the distance to the reference point is calculated (most commonly in the Euclidean metric).

$$d_i = \sqrt{\sum_j \left(z_{ij} - L_j\right)^2}$$

The determined distance is sufficient for establishing the ranking, although it is commonly practiced to calculate a measure expressed by the formula:

$$R_i = 1 - \frac{d_i}{\max_i d_i}$$

Currently, one of the most popular methods of linear ordering is TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution). It is a linear ordering method used in multi-criteria decision analysis. The normalization in this method can involve a transformation expressed by the formula:

$$z_{ij} = \frac{x_{ij}}{\sqrt{\sum_i x_{ij}^2}} \quad (5)$$

Next, the reference pattern and anti-pattern are determined according to the formulas:

$L_j^+ = \max_i z_{ij}$ for stimulant , $L_j^+ = \min_i z_{ij}$ for destimulant

and the anti-pattern according to the formulas:

$L_j^- = \max_i z_{ij}$ for destimulant, $L_j^- = \min_i z_{ij}$ for stimulant

In the next step, the distance to the reference pattern (d_i^+) and the anti-pattern (d_i^-) is calculated in the Euclidean metric.

The ranking is then performed based on the calculated coefficient of the relative proximity of decision variants to the ideal solution:

$$R_i = \frac{d_i^-}{d_i^- + d_i^+}$$

The higher the value of this coefficient, the better.

In the hereafter, functions which transform values of the criteria to the same scale (according to the convention adopted by (Harrington ,1965)). We will call it as desirability function. In particular, desirability function are transformations (1)-(5).

3. Selected properties of chosen linear ordering methods

Desirability function should ensure that variables are consistently scaled, comparisons across variables are meaningful and make sure the analysis is less susceptible to the distorting effects of extreme values. It is expected that rearranging the order of input variables or changing units e.g. from to thousands to millions has no impact on the ranking. Such properties shall be called the symmetry and scale-invariance, respectively. It also a natural property that the distorting effects should be under the control. To this end, the boundness property is desired.

Our goal is to investigate the aforementioned properties of the described desirability functions. These properties can be expressed in the language of functional equations and inequalities. Let $f: \mathbb{R}^n \to \mathbb{R}^n$.

1) Function f is symmetric
$$\sigma\big(f(x_1, \dots, x_n)\big) = f\big(x_{\sigma(1)}, \dots, x_{\sigma(n)}\big) \ for \ x_1, \dots, x_n \in \mathbb{R}$$
and for every permutation σ of the set $\{1, \dots, n\}$
2) Function f is scale-invariant
$$f(ax_1, \dots, ax_n) = f(x_1, \dots, x_n) \ for \ a \in (0, \infty), x_1, \dots, x_n \in \mathbb{R}$$
3) Function f is shift-invariant
$$f(x_1 + t, \dots, x_n + t) = f(x_1, \dots, x_n) \ \ for \ \ t \in (0, \infty),$$
$$x_1, \dots, x_n \in \mathbb{R}$$
4) Function f is bounded, that is there exist $m, M \in \mathbb{R}$ such that
$$m \le f(x_1, \dots, x_n) \le M \ \ for \ \ x_1, \dots, x_n \in \mathbb{R}$$

Let $x = (x_1, \dots, x_i, \dots, x_n)$ be a vector of a variable with n values, and x_i be the i-th value of this vector.

For the vector x, we will check whether properties 1)-4) hold for three desirability function of the form (1) which are part of the following ranking techniques: Simple Additive Ranking (SAR), Absolute reference and TOPSIS.

SAR method for the vector x uses the following desirability functions for standardizing features

$$f(x)=f(x_1, \dots, x_i, \dots, x_n)= \left(\frac{x_1 - \min(x)}{\max(x) - \min(x)}, \dots, \frac{x_i - \min(x)}{\max(x) - \min(x)}, \dots, \frac{x_n - \min(x)}{\max(x) - \min(x)}\right) \tag{6}$$

Firstly, we investigate a symmetry property.

Using properties of $\min(\cdot)$ and $\max(\cdot)$ function, for any permutation σ of $\{1, \dots, n\}$, the following equations hold:

$$\min(x) = \min(x_1, \dots, x_n) = \min\big(x_{\sigma(1)}, \dots, x_{\sigma(n)}\big)$$
$$\max(x) = \max(x_1, \dots, x_n) = \max\big(x_{\sigma(1)}, \dots, x_{\sigma(n)}\big)$$

First note that function f of the form (6) is affine transformation of x, that is $f(x) = ax + b$ with $a = \frac{1}{\max(x) - \min(x)} > 0$ and $b = \frac{-\min(x)}{\max(x) - \min(x)}$. Since such affine transformation preserves an order of x, we have

$$\sigma\big(f(x)\big) = \sigma\left(\frac{x_1 - \min(x)}{\max(x) - \min(x)}, \dots, \frac{x_n - \min(x)}{\max(x) - \min(x)}\right)$$

$$= \left(\frac{x_{\sigma(1)} - \min(x_\sigma)}{\max(x_\sigma) - \min(x_\sigma)}, \dots, \frac{x_{\sigma(n)} - \min(x_\sigma)}{\max(x_\sigma) - \min(x_\sigma)}\right) = f(x_\sigma)$$

is true for any permutation σ of $\{1, \dots, n\}$.

The next step is to check whether the function f is scale-invariant.

Fix $i \in \{1, \dots, n\}$. The i-th coordinate of the function $f(x_i)$ has a form

$$f(ax_i) = \frac{a \cdot x_i - \min(a \cdot x)}{\max(a \cdot x) - \min(a \cdot x)}$$

We can use the property of functions min(·) and max(·) such as :

$\min(a \cdot x) = a \cdot \min(x)$ and $\max(a \cdot x) = a \cdot \max(x)$

Therefore

$$f(ax_i) = \frac{a \cdot x_i - \min(a \cdot x)}{\max(a \cdot x) - \min(a \cdot x)} = \frac{a \cdot x_i - a \cdot \min(x)}{a \cdot \max(x) - a \cdot \min(x)}$$
$$= \frac{a \cdot (x_i - \min(x))}{a \cdot (\max(x) - \min(x))} = \frac{x_i - \min(x)}{\max(x) - \min(x)} = f(x_i)$$

In a result

$$f(ax) = f(ax_1, \dots, ax_i, \dots, ax_{n)} = f(x),$$

whence f is scale-invariant.

Next we examine shift-invariance. To this end, fix $t \in (0, \infty)$. Note that for fixed $i = 1, \dots n$, we have

$$f(x_i + t) = \frac{x_i + t - \min(x_i + t)}{\max(x_i + t) - \min(x_i + t)}$$

Using the property of functions min(·) and max(·) such as :

$$min(x + t) = min(x) + t$$

and

$$max(x + t) = max(x) + t$$

we obtain

$$f(x_i+t)=\frac{x_i+t-\min(x_i+t)}{\max(x+t)-\min(\mathrm{x}+t)}=\frac{x_i+t-\min(x_i)-t}{\max(\mathrm{x})+t-\min(\mathrm{x})-t}$$
$$=\frac{x_i-\min(\mathrm{x})}{\max(\mathrm{x})-\min(\mathrm{x})}=f(x_i).$$

Thus, the function f is shift-invariant.

Boundness is the last property to be checked. We will demonstrate that the lower bound and upper bound are given by $m=0$ and $M=1$, respectively. Observe that the denominator is a positive since x non-constant. As $min(x)\leq x_i\leq max(x)$ for $i\in\{1,\dots,n\}$, we have

$$0=\frac{\min(x)-\min(x)}{\max(x)-\min(x)}\leq\frac{x_i-\min(x)}{\max(x)-\min(x)}\leq\frac{\max(x)-\min(x)}{\max(x)-\min(x)}=1$$

The standardization method used in the Absolute Reference method is given by the formula:

$$f(x_1,\dots,x_n)=\left(\frac{x_1-\bar{x}}{S},\dots,\frac{x_n-\bar{x}}{S}\right) \tag{7}$$

where $\bar{x}$ is an arithmetic mean and S is a standard deviation of the variable $x=(x_1,\dots,x_n)$.

First, let us check the symmetry property. First observe that, for any permutation σ of $\{1,\dots,n\}$, it holds

$$\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}x_{\sigma(i)} \tag{8}$$

First note that function f of the form (7) is affine transformation of x, that is $f(x)=ax+b$ with $a=\frac{1}{S}>0$ and $b=\frac{-\bar{x}}{S}$. Since such affine transformation preserves an order of x , applying (8), we have

$$\sigma\big(f(x)\big)=\sigma\left(\frac{x_1-\bar{x}}{S},\dots,\frac{x_\mathrm{n}-\bar{x}}{S}\right)$$

$$= \left(\frac{x_{\sigma(1)} - \frac{1}{n} \cdot \sum_{i=1}^{n} x_{\boldsymbol{i}}}{\sqrt{\frac{\sum_{i=1}^{n} \left(x_{\boldsymbol{i}} - \frac{1}{n} \cdot \sum_{i=1}^{n} x_{\boldsymbol{i}} \right)^2}{n}}}, \dots, \frac{x_{\sigma(n)} - \frac{1}{n} \cdot \sum_{i=1}^{n} x_{\boldsymbol{i}}}{\sqrt{\frac{\sum_{i=1}^{n} \left(x_{\boldsymbol{i}} - \frac{1}{n} \cdot \sum_{i=1}^{n} x_{\boldsymbol{i}} \right)^2}{n}}} \right)$$

$$= \left(\frac{x_{\sigma(1)} - \frac{1}{n} \cdot \sum_{i=1}^{n} x_{\boldsymbol{\sigma}(i)}}{\sqrt{\frac{\sum_{i=1}^{n} \left(x_{\boldsymbol{\sigma}(i)} - \frac{1}{n} \cdot \sum_{i=1}^{n} x_{\boldsymbol{\sigma}(i)} \right)^2}{n}}}, \dots, \frac{x_{\sigma(n)} - \frac{1}{n} \cdot \sum_{i=1}^{n} x_{\boldsymbol{\sigma}(i)}}{\sqrt{\frac{\sum_{i=1}^{n} \left(x_{\boldsymbol{\sigma}(i)} - \frac{1}{n} \cdot \sum_{i=1}^{n} x_{\boldsymbol{\sigma}(i)} \right)^2}{n}}} \right)$$
$$= f(x_\sigma)$$

is true for any permutation σ of $\{1, \dots, n\}$. In a result the function f is symmetric. Next, we want to check if function f is scale-invariant. Fix $i \in \{1,2, \dots, n\}$. Then

$$f(a \cdot x_i) = \frac{a \cdot x_i - \frac{1}{n} \cdot \sum_{i=1}^{n} a \cdot x_i}{\sqrt{\frac{\sum_{i=1}^{n} \left(a \cdot x_i - \frac{1}{n} \cdot \sum_{i=1}^{n} a \cdot x_i \right)^2}{n}}} = \frac{a \cdot x_i - \frac{a}{n} \cdot \sum_{i=1}^{n} x_i}{\sqrt{\frac{\sum_{i=1}^{n} \left(a \cdot x_i - \frac{a}{n} \cdot \sum_{i=1}^{n} x_i \right)^2}{n}}}$$
$$= \frac{a \cdot (x_i - \frac{1}{n} \cdot \sum_{i=1}^{n} x_i)}{\sqrt{\frac{\sum_{i=1}^{n} \left(a(x_i - \frac{1}{n} \cdot \sum_{i=1}^{n} x_i) \right)^2}{n}}}$$
$$= \frac{a \cdot (x_i - \frac{1}{n} \cdot \sum_{i=1}^{n} x_i)}{\sqrt{\frac{\sum_{i=1}^{n} a^2 \left(x_i - \frac{1}{n} \cdot \sum_{i=1}^{n} x_i \right)^2}{n}}}$$
$$= \frac{a \cdot (x_i - \frac{1}{n} \cdot \sum_{i=1}^{n} x_i)}{\sqrt{\frac{a^2 \sum_{i=1}^{n} \left(x_i - \frac{1}{n} \cdot \sum_{i=1}^{n} x_i \right)^2}{n}}} = \frac{a \cdot (x_i - \frac{1}{n} \cdot \sum_{i=1}^{n} x_i)}{a \sqrt{\frac{\sum_{i=1}^{n} \left(x_i - \frac{1}{n} \cdot \sum_{i=1}^{n} x_i \right)^2}{n}}}$$
$$= f(x_i)$$

which proves that the function f is scale-invariant.

Next step is to verify whether desirability function is shift-invariant. Notice that for any $t > 0$, it holds

$$\sum_{i=1}^{n}(x_i + t) = n \cdot t + \sum_{i=1}^{n} x_i ,$$

Therefore

$$f(x_i + t) = \frac{x_i + t - \frac{1}{n} \cdot \sum_{i=1}^{n}(x_i + t)}{\sqrt{\frac{\sum_{i=1}^{n}\left(x_i + t - \frac{1}{n} \cdot \sum_{i=1}^{n}(x_i + t)\right)^2}{n}}}$$

$$= \frac{x_i + t - \frac{1}{n} \cdot (n \cdot t + \sum_{i=1}^{n}(x_i))}{\sqrt{\frac{\sum_{i=1}^{n}\left(x_i + t - \frac{1}{n} \cdot (n \cdot t + \sum_{i=1}^{n}(x_i))\right)^2}{n}}}$$

$$= \frac{x_i + t - t - \frac{1}{n}\sum_{i=1}^{n}(x_i))}{\sqrt{\frac{\sum_{i=1}^{n}\left(x_i + t - t - \frac{1}{n}\sum_{i=1}^{n}(x_i)\right)^2}{n}}}$$

$$= \frac{x_i - \frac{1}{n}\sum_{i=1}^{n}(x_i)}{\sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{i=1}^{n}(x_i)\right)^2}{n}}} = f(x_i)$$

which proves that the function f is shift-invariant.

Finally, we shall show that the function f does not have a property of boundness.

Fix $m \in \mathbb{R}$. Put $x = (x_1, x_2, .., x_n)$, such that $x_1 = x_2 = \cdots = x_{n-1} = 0$, but $x_n = 1$. Hence, $\bar{x} = \frac{1}{n}$ and $S = \sqrt{\frac{1-\frac{1}{n}}{n}}$. Let $n > m^2 + 1$. Then $\sqrt{n-1} > m$, which gives $\frac{n-1}{\sqrt{n-1}} > m$. Thus

$$\frac{\frac{n-1}{n}}{\frac{1}{n}\sqrt{n-1}} > m$$

and so

$$\frac{1 - \frac{1}{n}}{\sqrt{\frac{1}{n} - \frac{1}{n^2}}} > m.$$

In a result, we obtain

$$f(x_n) = \frac{1-\frac{1}{n}}{\sqrt{\frac{1-\frac{1}{n}}{n}}} > m.$$

In a consequence the function f in not bounded.

Desirability function in TOPSIS method is a transformation given by the formula

$$f(x) = f(x_1, \dots, x_i, \dots, x_n) = \left(\frac{x_1}{\sqrt{\sum_i x_i^2}}, \dots, \frac{x_i}{\sqrt{\sum_i x_i^2}}, \dots, \frac{x_n}{\sqrt{\sum_i x_i^2}}\right) \tag{9}$$

The first property we want to examine is the symmetry of the function f. First note that function f of the form (9) is linear transformation of x, that is $f(x) = ax$ with $a = \frac{1}{\sqrt{\sum_i x_i^2}} > 0$. Since such linear transformation preserves an order of x, applying (8), we have

$$\sigma\left(f(x)\right) = \sigma\left(\frac{x_1}{\sqrt{\sum_i x_i^2}}, \dots, \frac{x_n}{\sqrt{\sum_i x_i^2}}\right) = \left(\frac{x_{\sigma(1)}}{\sqrt{\sum_i x_i^2}}, \dots, \frac{x_{\sigma(n)}}{\sqrt{\sum_i x_i^2}}\right)$$

$$= \left(\frac{x_{\sigma(1)}}{\sqrt{\sum_i x_{\sigma(i)}^2}}, \dots, \frac{x_{\sigma(n)}}{\sqrt{\sum_i x_{\sigma(i)}^2}}\right) = f\left(x_\sigma\right)$$

Therefore, function f is symmetric.

Secondly, we need to check if the *f* function is scale-invariant. For fixed $i \in \{1, \dots, n\}$, we get

$$f(a \cdot x_i) = \frac{a \cdot x_i}{\sqrt{\sum_i (a \cdot x_i)^2}} = \frac{a \cdot x_i}{\sqrt{\sum_i a^2 \cdot {x_i}^2}} = \frac{a \cdot x_i}{a \cdot \sqrt{\sum_i {x_i}^2}} = \frac{x_i}{\sqrt{\sum_i {x_i}^2}} = f(x_i)$$

That proves the scale-invariance of the desirability function.

Next, we shall that the shift-invariance property does not hold.

Notice that for any $i = 1, \dots, n$, we have

$$f(x_i + t) = \frac{x_i + t}{\sqrt{\sum_i (x_i + t)^2}} = \frac{x_i + t}{\|x + t\|}$$

To this end, let us show first that for any positive numbers a, b and k, where $a < b$, the following inequality holds

$$\frac{a+k}{b+k} > \frac{a}{b} \tag{10}$$

Since $a < b$ and $k > 0$, we get $ak < bk$. Hence

$$b(a + k) = ab + bk > ab + ak = a(b + k)$$

Dividing both sides of above inequality by $b(b + k)$, we obtain (10). Define $\|x\| = \sqrt{\sum_i x_i^2}$. Since $\|\cdot\|$ is the norm, it holds $\|x + t\| < \|x\| + \|t\|$ for any $x \neq 0$ and $t \in (0, \infty)$. Hence, using (18), we can observe that

$$f(x_i + t) = \frac{x_i + t}{\|x + t\|} > \frac{x_i + t}{\|x\| + t} > \frac{x_i}{\|x\|} = f(x_i)$$

for $i = 1, \dots, n$. Thus, the function f does not have a shift-invariance property.

The boundedness of the function can be proven by starting with the obvious inequality

$$0 \leq x_i^2 \leq \sum_i x_i^2$$

for $i = 1, \dots, n$. And so

$$-\sqrt{\sum_i x_i^2} \leq x_i \leq \sqrt{\sum_i x_i^2}$$

hence

$$-1 \leq \frac{x_i}{\sqrt{\sum_i x_i^2}} \leq 1.$$

In this way we proved that the lower bound and upper bound are given by $m = -1$ and $M = 1$, respectively, which proves the boundedness of the function f.

4. Conclusions

Incorporating the functional equations and inequalities enabled to express the very intuitive and desired properties of ranking techniques, namely: symmetry, scale-invariance, shift-invariance, and boundness in a clear and neat way. Straightforward proofs revealed which properties hold in a case of three investigated ranking techniques. In a result, a decision-maker can evaluate each of them and choose the optimal one for a given problem.

Investigating the properties of ranking techniques is essential to ensure the consistency of data, compatibility with analysis algorithms, meaningful comparisons, robustness against outliers, interpretability of results, avoidance of biases, and effective communication of findings in the context of data analysis. It is a critical step toward conducting reliable and valid analyses.

References

Brüggemann, R., Simon, U., Mey, S., Estimation of averaged ranks by extended local partial order models, (2005), Commun. Math. Comput. Chem. Vol. 54, 489-518.

Figueira, J., Greco, J., Ehrgott, M., (2005), Multiple Criteria Decision Analysis: State of the Art Surveys, International Series in Operations Research & Management Science, Vol. 78).

Harrington, E.C., (1965). The Desirability Function. Industrial Quality Control., 21, 494-498.

Hellwig, Z., (1965) Application of the Taxonomic Method to the Typological Division of Countries According to the Level of Their Development and the Resources and Structure of Qualified Personnel. Przegląd Statystyczny 1968, 4, 307–326.

Morgenstern O., von Neumann J., (1995), Theory of Games and Economic Behavior, Princeton University Press.

Quiggin J., (1982), A theory of anticipated utility. Journal of Economic Behavior and Organization, 3, 323–343.

Kahneman, D., Tversky, A. (1979). Prospect Theory: An Analysis of Decision under Risk. Econometrica, 47(2), 263-292

Tversky, A., Kahneman D., (1992), Advances in prospect theory: cumulative representation of uncertainty, Journal of Risk and Uncertainty, 5, 297–323.

Schmeidler, D., (1989), Subjective probability and expected utility without additivity. Econometrica 57, 571–587.

Prelec, D., (1998),The Probability Weighting Function, Econometrica, 66, 497–527.

Atkinson, A.B., (1970) On the measurement of inequality. Journal of Economic Theory, 2 (3), 244–263, doi:10.1016/0022-0531(70)90039-6

Roy, B. (1996), Multicriteria Methodology for Decision Aiding.

Saaty, T. L., (1994), Fundamentals of Decision Making and Priority Theory with the Analytic Hierarchy Process.

Von Winterfeldt, D., & Edwards, W., (1986). Decision Analysis and Behavioral Research, Cambridge University Press.

Zeleny, M., (1982), Multiple Criteria Decision Making, McGraw-Hill, New York